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Strongly Unique Minimal Projections on Hyperplanes*

MARCO BARONTI

Istituto di Matematica, Facolta' di Ingegneria, Piazzale Kennedy, Pad. D, Genova, Italy

AND

GRZEGORZ LEWICKI

Department of Mathematics, Uniwersytet Jagiellonski, 30-059 Krakow, Reymonta 4, Poland

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G. Lewicki (*J. Approx. Theory* **64** (1991), 181–202) studied strongly unique minimal projections in reflexive Banach spaces and in l^n_∞ he obtained a complete characterization of those hyperplanes that are the range of a strongly unique minimal projection. In this paper we extend this type of characterization to hyperplanes in l_∞ and l^n_1.

0. Introduction

Let X be a real Banach space and Y a proper subspace of X. A bounded, linear map $P: X \to Y$ is called a projection if and only if: $Py = y$ for any $y \in Y$. Obviously, if $Y \neq \{0\}$, then $\|P\| \geqslant 1$ for any projection P. The set of all projections going from X onto Y will be denoted by $P(X, Y)$. Set $\lambda(Y, X) = \inf\{\|P\|; P \in P(X, Y)\}$. A projection P is *minimal* if $\|P\| = \lambda(Y, X)$. The study of existence and unicity of minimal projections is related to the study of best approximation.

In this paper we would like to investigate strong unicity of minimal projections on hyperplanes of l_∞ and l^n_1. Recall that, given a Banach space B and $D \subset B$, $D \neq \phi$, an element $y \in D$ is called a strongly unique best approximation (briefly SUBA) to $x \in B$ if and only if for every $d \in D$,

$$\|x - d\| \geqslant \|x - y\| + r \, \|y - d\|, \tag{0.1}$$

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where the constant $r > 0$ is independent of $d \in D$. Note that strong unicity yields the continuity of the metric projection (see [6, p. 82])

> $x \in B \to P_D(x) \in D$, where $P_D(x)$ denotes the fixed element from the set of all best approximants of x in D.

In the case of projections, condition (0,1) suggests the following:

DEFINITION 0.1 (See [8]). Let X be a real Banach space and let Y be a proper subspace of X. An operator $P_0 \in P(X, Y)$ is called a strongly unique minimal projection (briefly a SUM projection) if and only if there is $r \in (0, 1]$ such that $\|P\| \geqslant \|P_0\| + r\,\|P - P_0\|$ for any $P \in P(X, Y)$.

In Section 1 we present a complete characterization of those hyperplanes in l_∞ which are the range of a SUM projection. In Section 2 we will be concerned with the case of l_1^n. In the sequel some preliminary results will be needed. We start with

LEMMA 0.2 (See [4]). *To each $f \in l_\infty^*$ there corresponds a unique element $h \in l_1$ such that $f - h \in c_0^\perp$. Furthermore,* $\|f\| = \|h\| + \|f - h\|$.

LEMMA 0.3 (See [4, Thm. 1.4]). *Let f and h be as described in Lemma 0.2. In order that $\lambda(f^{-1}(0), l_\infty) = 1$, it is necessary and sufficient that $\|f\| \leqslant 2\,\|h\|_\infty$.*

THEOREM 0.4 (See [4, Thm. 2.4]). *Let f and h be as described in Lemma 0.2 and suppose that $1 = \|f\| > 2\,\|h\|_\infty$. Then*

$$\lambda(f^{-1}(0), l_\infty) = 1 + \left[\|f - h\| + \sum_{i=1}^{+\infty} |h_i|\,(1 - 2\,|h_i|)^{-1}\right]^{-1}.$$

THEOREM 0.5 (See [4]). *Let f and h be as in Theorem 0.4. $f^{-1}(0)$ has a minimal projection if and only if there exists an $x \in l_\infty$ such that $\|x\| = 1$, $|h(x)| = \|h\|$, and $|(f - h)(x)| = \|f - h\|$.*

By the proof of Theorem 0.3 in [4] it is easy to prove

THEOREM 0.6. *Let f and h be as in Theorem 0.4. Then there is a unique norm-one projection if and only if $\|h\| + \|f - h\| \leqslant 2\,|h_{i_0}|$ for a unique index i_0.*

If $\lambda(f^{-1}(0), l_\infty) > 1$, then there is a unique minimal projection P_0 if and only if $h_i \neq 0$ for any $i \in N$. The projection P_0 is given by

$$P_0 x = x - f(x)\, z^0, \qquad \text{where} \quad z^0, x \in l_\infty,$$

and

$$z_n^0 = \frac{(\lambda(f^{-1}(0), l_\infty) - 1) \operatorname{sgn} h_n}{1 - 2|h_n|} \qquad \text{for any} \quad n \in N.$$

PROPOSITION 0.7 (See [3, 4]). *Let f and h be as in Theorem 0.4. Let $P \in P(f^{-1}(0), l_\infty)$, $Px = x - f(x)\, z$, where $z \in f^{-1}(1)$. Then*

$$\|P\| = \sup_{n \in N} \{ |1 - h_n z_n| + |z_n| (1 - |h_n|) \}.$$

For more complete information about minimal projections on subspaces of finite codimension in l_p, $1 \leqslant p \leqslant \infty$, the reader is referred to [1–5]. Now assume Y is a proper finite dimensional subspace of a real Banach space X. Let $S(X)$ be the unit sphere in X and ext (X) the set of its extremal points. For $P \in P(X, Y)$ put

$$\operatorname{crit} P = \{ f \in \operatorname{ext}(X^*) \colon \|f \circ P\| = \|P\| \}. \tag{0.2}$$

By [8, Lemma 2.1], we get that for every $P \in P(X, Y)$ crit P is a nonempty set. By Theorem 2.3 from [8] it is easy to deduce the following:

THEOREM 0.8. *Assume X is a reflexive space and let $Y \subset X$ be one of its finite dimensional subspaces. For given $P_0 \in P(X, Y)$ and $f \in \operatorname{crit} P_0$ put*

$$A_f = \{ x \in \operatorname{ext}(X) \colon f(P_0 x) = \|P_0\| \}. \tag{0.3}$$

Then we have

(a) *P_0 is a minimal projection if and only if for every $P \in P(X, Y)$ there exists $f \in \operatorname{crit} P_0$ such that*

$$\inf \{ f(P - P_0) x \colon x \in A_f \} \leqslant 0. \tag{0.4}$$

(b) *P_0 is a SUM projection with a constant $r > 0$ if and only if for every $P \in P(X, Y)$ there exists $f \in \operatorname{crit} P_0$ such that*

$$\inf \{ f(P - P_0) x \colon x \in A_f \} \leqslant -r\, \|P - P_0\|. \tag{0.5}$$

Remark 0.9. In Theorem 0.5 the set crit P_0 may be replaced by any set $C \subset \operatorname{crit} P_0$ such that $C \cup -C = \operatorname{crit} P_0$ and $C \cap -C = \phi$.

Remark 0.10 (See [5]). Let X be a Banach space and let $Y \subset X$ be a closed hyperplane. Then for each $P \in P(X, Y)$ there exists a unique $y^P \in X$ with $f(y^P) = 1$ ($Y = \ker f$, $f \in X^*$, $\|f\| = 1$) such that $Px = x - f(x)\, y^P$ for every $x \in X$.

Let us concentrate on this case: $X = l_1^n$ and $Y = \ker f$ for some $f \in S(l_\infty^n)$.

Remark 0.11 (See [5]). For every $P \in P(X, Y)$,

$$\|P\| = \max_{i=1,\dots,n} \|Pe_i\|$$

and

$$\|Pe_i\| = |1 - f_i y_i^p| + |f_i| \, (\|y^p\| - |y_i^p|).$$

PROPOSITION 0.12 (See [5]). *Let $P_0 \in P(X, Y)$ be a minimal projection. Then $\|P_0\| = 1$ if and only if a functional f corresponding to Y has at most two coordinates different from 0. There exists a unique projection P_0 of norm one if and only if exactly two coordinates of f are different from 0.*

THEOREM 0.13 (See [5, 9]). *Assume $f \in S(l_\infty^n)$; $f = (1, f_2, \dots, f_n)$, $1 \geqslant f_2 \geqslant \cdots \geqslant f_n \geqslant 0$, $f_3 > 0$. For $i, j \geqslant 3$ let us set*

$$a_i = \sum_{j=1}^{i} f_j, \qquad b_i = \sum_{j=1}^{i} f_j^{-1}, \qquad \beta_i = b_i/(i-2), \tag{0.6}$$

and

$$c_j = \min\{f_j b_{j-1}, a_{j-1}\}. \tag{0.7}$$

Put

$$i = i(f) = \max\{j \geqslant 3 \colon c_j \geqslant j - 3\}. \tag{0.8}$$

If $P_0 \in P(X, Y)$ is a minimal projection then $\|P_0\| = 1 + v$, where

$$v = \begin{cases} 2((\beta_i - f_i^{-1})(i-2) + a_i f_i^{-1} - i)^{-1} & \text{if} \quad a_i < i - 2 \\ 2(a_i \beta_i - i)^{-1} & \text{if} \quad a_i \geqslant i - 2. \end{cases} \tag{0.9}$$

Moreover, if $a_i \geqslant i - 2$ the vector y^0 corresponding to P_0 has coordinates

$$y_1^0 = v(\beta_i - f_i^{-1})/2, \dots, \qquad y_i^0 = v(\beta_i - f_i^{-1})/2, \qquad y_k^0 = 0 \quad \text{for} \quad k = i+1, \dots, n \tag{0.10}$$

If $a_i < i - 2$, then

$$\begin{aligned} y_1^0 &= v((i-2)(\beta_i - f_i^{-1}) + f_i^{-1} - 1)/2 \\ y_k^0 &= v(f_i^{-1} - f_k^{-1})/2 \qquad \text{for} \quad k = 2, \dots, i \\ y_k^0 &= 0 \qquad \text{for} \quad k = i+1, \dots, n. \end{aligned} \tag{0.11}$$

COROLLARY 0.14 (See [9]). *Let* $f \in S(l^n_\infty)$ *be as in Theorem* 0.13. *Put*

$$u = u(f) = \begin{cases} i(f) & \text{if } \beta_{i(f)} > f^{-1}_{i(f)} \\ m(f) & \text{if } \beta_{i(f)} = f^{-1}_{i(f)}, \end{cases} \tag{0.12}$$

where

$$m(f) = \min\{ j \leqslant i(f) - 1 : f_{j+1} = f_{i(f)} \}. \tag{0.13}$$

If $a_u \geqslant u - 2$ (*resp.* $a_u < u - 2$) *then the formula* (0.10) (*resp.* (0.11)) *defines for* $i = u$ *the coordinates of the vector* y^0 *corresponding to* P_0.

⟨1⟩

First we state a lemma that will be of use later.

LEMMA 1.1. *Let* $f \in S(l^*_\infty)$ *and* $h \in l_1$ *be as in Lemma* 0.2. *If* $1 \leqslant 2\,|h_{i_0}|$ *for a unique* $i_0 \in N$ *and if there is* $y \in f^{-1}(0)$ *such that* $\|y\| = |y_{i_0}| > |y_n|$ *for any* $n \neq i_0$ *then* $h_n = 0$ *for* $n \neq i_0$ *and* $|h_{i_0}| = \|f - h\| = 1/2$.

Proof. Note that $(2\,|h_{i_0}| - \|h\|)\,\|y\|_\infty \leqslant |h_{i_0}|\,|y_{i_0}| - \sum_{n \neq i_0} |h_n|\,|y_n| \leqslant |h(y)| = |(f-h)(y)| \leqslant \|f - h\|\,\|y\|_\infty \leqslant (2\,|h_{i_0}| - \|h\|)\,\|y\|_\infty$. Since $|y_n| < \|y\|_\infty$ for any $n \neq i_0$, we have $h_n = 0$ for any $n \neq i_0$. So $\|h\| = |h_{i_0}| = \|f - h\| = 1/2$.

Now we will prove the main result of this section.

THEOREM 1.2. *Let* $f \in S(l^*_\infty)$ *and* $h \in l_1$ *be as in Lemma* 0.2. *Then there is a SUM projection onto* $Y = f^{-1}(0)$ *if and only if* Y *is the range of exactly one norm-one projection, i.e.,* $1 \leqslant 2\,|h_{i_0}|$ *for a unique* $i_0 \in N$. (*Compare with Theorem* 0.6.)

Proof. "If" part. Assume $1 = \|h\| + \|f + h\| \leqslant 2\,|h_{i_0}|$ for a unique $i_0 \in N$. If we define $z^0 = (1/h_{i_0})e_{i_0}$ then we have $f(z^0) = h(z^0) = 1$. Put $P_0 x = x - f(x) z^0$. Applying Proposition 0.7, it is easy to verify that $\|P_0\| = 1$. We will show that P_0 is a SUM projection. To do this, take any $P \in P(l_\infty, Y)$. According to Remark 0.10, there is $z \in f^{-1}(1)$ such that $Px = x - f(x)z$ for any $x \in l_\infty$. Now we divide the proof into three cases.

Case 1. $\|z - z^0\|_\infty = |z_{i_0} - z^0_{i_0}| > |z_n - z^0_n|$ for any $n \neq i_0$. Observe that $z - z^0 \in Y$ and by Lemma 1.1 we have: $h = (1/2)\operatorname{sgn}(h_{i_0})e_{i_0}$. Consequently $z^0 = 2\operatorname{sgn}(h_{i_0})e_{i_0}$. Since $\|z - z^0\|_\infty = |z_{i_0} - z^0_{i_0}| = |z_{i_0} - 2\operatorname{sgn}(h_{i_0})| > |z_n - z^0_n| = |z_n|$ for any $n \neq i_0$, we have $|z_{i_0} - 2\operatorname{sgn}(h_{i_0})| \geqslant \sup_{n \neq i_0} |z_n|$. According to Proposition 0.7, $\|P\| = \max\{|1 - h_{i_0} z_{i_0}| + |z_{i_0}|\,(1 - |h_{i_0}|);\ 1 + \sup_{n \neq i_0} |z_n|\} \leqslant \max\{|1 - h_{i_0} z_{i_0}| + |z_{i_0}|\,(1 - |h_{i_0}|);\ 1 + |z_{i_0} - 2\operatorname{sgn}(h_{i_0})|\}$.

Set $w = z - z_{i_0} e_{i_0}$. Then $|(f-h)(w)| = |(f-h)(z)| = |f(z) - h(z)| = |1 - h_{i_0} z_{i_0}| = |1 - (1/2)\,\mathrm{sgn}(h_{i_0}) z_{i_0}| \leqslant \|f - h\|\,\|w\| = (1/2)\sup_{n \neq i_0} |z_n|$ and consequently $\sup_{n \neq i_0} |z_n| \geqslant |2 - \mathrm{sgn}(h_{i_0}) z_{i_0}| = |z_{i_0} - 2\,\mathrm{sgn}(h_{i_0})|$. Hence $\|P\| = \max\{|1 - h_{i_0} z_{i_0}| + |z_{i_0}|\,(1 - |h_{i_0}|);\ 1 + |z_{i_0} - 2\,\mathrm{sgn}(h_{i_0})|\} = \max\{|1 - (1/2)\,\mathrm{sgn}(h_{i_0}) z_{i_0}| + |z_{i_0}|/2;\ 1 + |z_{i_0} - 2\,\mathrm{sgn}(h_{i_0})|\} = \max\{(|2\,\mathrm{sgn}(h_{i_0}) - z_{i_0}| + |z_{i_0}|)/2;\ 1 + |z_{i_0} - 2\,\mathrm{sgn}(h_{i_0})|\}$. Note that $(|2\,\mathrm{sgn}(h_{i_0}) - z_{i_0}| + |z_{i_0}|)/2 < 1 + |z_{i_0} - 2\,\mathrm{sgn}(h_{i_0})|$ since $|z_{i_0}| - 2 \leqslant |z_{i_0} - 2\,\mathrm{sgn}(h_{i_0})|$.

From this, it follows that

$$\|P\| = 1 + |z_{i_0} - 2\,\mathrm{sgn}(h_{i_0})|$$

$$\|P - P_0\| = \|z - z^0\| = |z_{i_0} - 2\,\mathrm{sgn}(h_{i_0})|$$

and so

$$\|P\| = \|P_0\| + r\,\|P - P_0\|,$$

where $r = 1 = \min_{n \neq i_0}\{1 - 2\,|h_n|\}$.

Case 2. $\|z - z^0\|_\infty = |z_{i_0} - z^0_{i_0}| = |z_{n_0} - z^0_{n_0}|$ for some $n_0 \neq i_0$. It is clear that $\|P\| \geqslant |1 - h_{n_0} z_{n_0}| + |z_{n_0}|\,(1 - |h_{n_0}|) \geqslant 1 + |z_{n_0}|\,(1 - 2\,|h_{n_0}|)$. Since $z^0 = (1/h_{i_0}) e_{i_0}$ and $n_0 \neq i_0$ we have $\|P\| \geqslant 1 + |z_{n_0} - z^0_{n_0}|\,(1 - 2\,|h_{n_0}|) = \|P_0\| + (1 - 2\,|h_{n_0}|)\,\|P - P_0\| \geqslant \|P_0\| + \min_{n \neq i_0}\{1 - 2\,|h_n|\}\,\|P - P_0\|$.

Case 3. $\|z - z^0\|_\infty > |z_{i_0} - z^0_{i_0}|$. Let ε be greater than 0. Then there is an index $n_\varepsilon \neq i_0$ such that $\|z - z^0\|_\infty < |z_{n_\varepsilon} - z^0_{n_\varepsilon}| + \varepsilon = |z_{n_\varepsilon}| + \varepsilon$. Hence $\|P\| \geqslant 1 + |z_{n_\varepsilon}|\,(1 - 2\,|h_{n_\varepsilon}|) > 1 + (1 - 2\,|h_{n_\varepsilon}|)(\|z - z^0\|_\infty - \varepsilon) \geqslant 1 + \min_{n \neq i_0}\{1 - 2\,|h_n|\}\,(\|P - P_0\| - \varepsilon)$. Passing with ε to 0 we obtain

$$\|P\| \geqslant 1 + \min_{n \neq i_0}\{1 - 2\,|h_n|\}\,\|P - P_0\|.$$

"Only if" part. Let P_0 be a SUM projection. So there is $r > 0$ such that $\|P\| \geqslant \|P_0\| + r\,\|P - P_0\|$ for any projection P. Obviously P_0 is the unique minimal projection, in fact if $\|P\| = \|P_0\|$ we have $r\,\|P - P_0\| = 0$ and since $r > 0$ we obtain $P = P_0$.

If there is a SUM projection of norm-one, then by Theorem 0.6 and by above argument we come to this thesis. Let $P_0 x = x - f(x) z^0$ be a SUM projection, $\|P_0\| > 1$, so suppose $1 > 2\,|h_n|$ for any $n \in N$. Then P_0 is exactly one minimal projection and, by Theorem 0.6, we have: $h_n \neq 0$ for any $n \in N$. We can suppose without loss of generality that

Case 1, there is $\mu \in N$: $h_n > 0$ for any $n > \mu$

or

Case 2, for any $\mu \in N$ there is $n > \mu$ such that $h_n > 0$.

(In fact if there is μ_0 such that $h_n < 0$ for any $n > \mu_0$ it is sufficient to consider $-f$ to return to case 1.) Denote

$$\alpha_n = |1 - h_1(z_1^0 + (\operatorname{sgn} h_1) h_n)| + |z_1^0 + (\operatorname{sgn} h_1) h_n| (1 - |h_1|)$$

$$\beta_n = |1 - h_n(z_n^0 - (\operatorname{sgn} h_1) h_1)| + |z_n^0 - (\operatorname{sgn} h_1) h_1 |(1 - |h_n|).$$

Since $\alpha_n \to |1 - h_1 z_1^0| + |z_1^0| (1 - |h_1|)$ there is $\nu \in N$ such that $\alpha_n < |1 - h_1 z_1^0| + |z_1^0| (1 - |h_1|) + r |h_1|/2$ for any $n > \nu$. Put $y_n = (\operatorname{sgn} h_1) [h_n e_1 - h_1 e_n] \in Y$, $z_n = z^0 + y_n$ and $P_n x = x - f(x) z_n$. Note that $z_n \neq z^0$ and so $\|P_n\| > \|P_0\|$. Consequently $\|P_n\| = \sup_k \{|1 - h_k z_k| + |z_k| (1 - |h_k|)\} = \max\{\alpha_n, \beta_n, \sup_{k \neq 1, n}\{|1 - h_k z_k^0| + |z_k^0| (1 - |h_k|)\}\} = \max\{\alpha_n, \beta_n\}$. (in fact $\sup_{k \neq 1, n}\{|1 - h_k z_k^0| + |z_k^0| (1 - |h_k|)\} \leqslant \|P_0\| < \|P_n\|$). Let $\alpha_n > \beta_n$. Then $\|P_n\| = \alpha_n < |1 - h_1 z_1^0| + |z_1^0| (1 - |h_1|) + r |h_1|/2 \leqslant \|P_0\| + r |h_1|/2$. But $\|P_n\| \geqslant \|P_0\| + r \|P_n - P_0\| = \|P_0\| + r \|z_n - z^0\| = \|P_0\| + r \|y_n\| \geqslant \|P_0\| + r |h_1|$. By the fact $h_1 \neq 0$ and $r > 0$ we obtain a contradiction which implies $\alpha_n \leqslant \beta_n$ So $\|P_n\| = \beta_n$. Hence $|1 - h_n(z_n^0 - (\operatorname{sgn} h_1) h_1)| + |z_n^0 - (\operatorname{sgn} h_1) h_1| (1 - |h_n|) \geqslant \|P_0\| + r \|P_n - P_0\| = \|P_0\| + r \|z_n - z^0\|$.

We recall that $\|P_0\| = \lambda(Y, l_\infty)$, $\|P_n - P_0\| = \|z_n - z^0\|_\infty = \|y_n\|_\infty \geqslant |h_1|$, $z_n^0 = (\|P_0\| - 1) \operatorname{sgn}(h_n)/(1 - 2|h_n|)$. Assume Case 1: $h_n > 0$ for $n > \mu$. Then we have $z_n^0 \to (\|P_0\| - 1) \equiv \gamma$, hence: $1 + |\gamma - |h_1|| \geqslant 1 + \gamma + r |h_1|$. Since $|h_1| > 0$, it implies $2\gamma \leqslant |h_1| (1 - r)$. Assume case 2: $h_{n_k} > 0$ $n_k < n_{k+1}$ for any k. So $z_{n_k}^0 \to \gamma$ and again we have $2\gamma \leqslant |h_1| (1 - r)$. If we repeat the same argument for $h_2, h_3, \ldots$ we obtain: $2\gamma \leqslant |h_n| (1 - r)$ for any $n \in N$ and consequently $\gamma = 0$. This contradiction completes the proof of Theorem 1.2. ∎

Remark 1.3. We point out that no SUM projection can exist on hyperplanes of c_0. In fact, in this case the minimal projections are not unique (see [5]).

Note that there exist hyperplanes in l_∞^n which posses SUM projections of norm greater than one because of

THEOREM 1.4 (See [8]). *Let $Y \subset l_\infty^n$ be a hyperplane i.e.: $Y = f^{-1}(0)$ for some $f = (f_1, \ldots, f_n)$ in l_1^n such that $\|f\| = 1$. Assume that P_0 is a minimal projection. Then P_0 is a SUM projection of norm greater than one if and only if $0 < 2|f_i| < 1$ for any i.*

⟨2⟩

In this section we consider the case $X = l_1^n$, $Y = \ker f$, where $f = (f_1, f_2, \ldots, f_n) \in S(l_\infty^n)$. According to Remark 0.11 we may assume without loss of generality that

$$1 = f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n \geqslant 0.$$

First we consider the simple case, when the norm of minimal projection is equal to one.

PROPOSITION 2.1. *Let $P_0 \in P(X, Y)$ be a minimal projection, $\|P_0\| = 1$. Then P_0 is a unique minimal projection if and only if P_0 is a SUM projection.*

Proof. We may assume $1 = f_1 \geq f_2 > 0 = \cdots = f_n$. It is easy to verify that if we put $y_1 = y_2 = 1/(f_1 + f_2)$ and $y_3 = \cdots = y_n = 0$, the projection P_y induced by $y = (y_1, ..., y_n)$ is a minimal projection. In order to prove that $P_0 = P_y$ is a SUM projection, we take an arbitrary $P \in P(X, Y)$ and write P in the form $P = I - f(\cdot)\, y^P$. It is clear that $\|P - P_0\| = \|y - y^P\|_1$. If $y_1^P < 0$ then

$$\|y - y^P\|_1 = \left\|\left(y_1^P - \frac{1}{f_1 + f_2}, y_2^P - \frac{1}{f_1 + f_2}, y_3^P, ..., y_n^P\right)\right\|_1$$

$$= \sum_{i=1}^{n} |y_i^P| = \|y^P\|_1.$$

Hence, by Remark 0.11, $\|P\| \geq \|Pe_1\| = |1 - y_1^P| + \|y^P\|_1 - |y_1^P| = 1 - y_1^P + \|y^P\|_1 + y_1^P = 1 + \|y^P\|_1 \geq \|P_0\| + f_2\, \|P - P_0\|$. If $y_2^P < 0$, by the same reasoning, we get $\|P\| \geq \|P_0\| + f_2\, \|P - P_0\|$.

Now we suppose $y_1^P > y_2^P > 0$. It is evident that in this case $\|y - y^P\|_1 = \|y^P\|_1 - 2\, |y_2^P|$, since $y_1^P + f_2\, y_2^P = 1$. Observe that

$$\begin{aligned} \|P\| \geq \|Pe_2\| &= |1 - f_2\, y_2^P| + f_2(\|y^P\|_1 - |y_2^P|) \\ &= 1 + f_2(\|y^P\|_1 - 2\, |y_2^P|) = \|P_0\| + f_2\, \|y - y^P\|_1 \\ &= \|P_0\| + f_2\, \|P - P_0\|. \end{aligned} \tag{2.1}$$

If $y_2^P > y_1^P > 0$, calculating as in the previous situation, we get the desired result. Since if P_0 is a SUM projection, it must be a unique minimal projection, the proof is complete. ∎

REMARK 2.2. *The constant f_2 obtained in proving Proposition 2.1 is the best possible.*

Proof. Take $y \in S(l_1^n)$ such that $f(y) = 1$ and $y_1 > y_2 > 0$. Let $P_y \in P(X, Y)$ be a projection induced by y. Note that $\|P_y e_1\| = |1 - y_1| + \|y\|_1 - |y_1| = 1 + \|y\|_1 - 2\, y_1 < 1$. Hence $\|P_y\| = \|P_y e_2\|$, since $\|P_y\| > 1$ and $\|P_y e_i\| = 1$ for $i \geq 3$. Following (2.1) the proof is complete. ∎

Now we will investigate the most difficult case, in which a norm of minimal projection is greater than one. Following Remark 0.11 and Proposition 0.12, we may assume without loss of generality that

$$1 = f_1 \geq f_2 \geq \cdots f_n \geq 0, \quad f_3 > 0, \qquad n \geq 3. \tag{2.2}$$

First let us prove some preliminary results

LEMMA 2.3. *Let f satisfy (2.2). If for $m \in \{3, ..., n\}$ $a_m > m-2$ there exists $y \in Ker\, f \setminus \{0\}$ satisfying the system of inequalities*

$$y_j \geqslant \sum_{\substack{i=1 \\ i \neq j}}^{n} y_i + \sum_{i=1}^{n-m} |y_{i+m}| \qquad \textit{for} \quad j = 1, 2, ..., m, \tag{2.3}$$

then there is $y^1 \in \operatorname{Ker} f \setminus \{0\}$ with

$$y_j^1 > \sum_{\substack{i=1 \\ i \neq j}}^{m} y_i^1 + \sum_{i=1}^{n-m} |y_{i+m}^1| \qquad \textit{for} \quad j = 1, 2, ..., m \tag{2.4}$$

(we define $\sum_{i=1}^{n-n} |y_{i+n}| = 0$).

Proof. Take $y \in \operatorname{Ker} f \setminus \{0\}$ satisfying (2.3).

Case 1. There exists $j \in \{1, 2, ..., m\}$ with

$$y_j > \sum_{\substack{i=1 \\ i \neq j}}^{n} y_i + \sum_{i=1}^{n-m} |y_{i+m}|.$$

Then we can find $\theta > 0$ with

$$y_j - \theta > \sum_{\substack{i=1 \\ i \neq j}}^{n} y_i + \sum_{i=1}^{n-m} |y_{i+m}| + (m-1)\theta \frac{f_j}{(a_m - f_j)}.$$

Let $y_j^1 = y_j - \theta$, $y_i^1 = y_i + \theta(f_j/(a_m - f_j))$ for $i \in \{1, 2, ..., m\} \setminus \{j\}$, $y_i^1 = y_i$ for $i \in \{m+1, ..., n\}$ and put $y^1 = (y_1^1, ..., y_n^1)$. Observe that

$$\sum_{i=1}^{n} f_i y_i^1 = \sum_{i=1}^{m} f_i y_i^1 + \sum_{i=m+1}^{n} f_i y_i = f_j y_j - f_j \theta$$
$$+ \sum_{\substack{i=1 \\ i \neq j}}^{m} f_i \left(y_i + \frac{\theta f_j}{a_m - f_j} \right) + \sum_{i=m+1}^{n} f_i y_i = \sum_{i=1}^{n} f_i y_i = 0.$$

To finish the proof, take $i \in \{1, 2, ..., m\} \setminus \{j\}$. Since $a_m > m-2$, then $a_m - f_j > f_j(m-3)$ which gives

$$\theta \frac{f_j}{(a_m - f_j)} > (m-2)\theta \frac{f_j}{(a_m - f_j)} - \theta. \tag{2.5}$$

Combining (2.3) with (2.5) we obtain

$$y_i^1 > \sum_{\substack{k=1 \\ k \neq i}}^{m} y_k^1 + \sum_{k=1}^{n-m} |y_{k+m}^1|,$$

which establishes formula (2.4).

Case 2.

$$y_j = \sum_{\substack{i=1 \\ i \neq j}}^{m} y_i + \sum_{i=1}^{n-m} |y_{i+m}| \qquad \text{for} \quad j = 1, 2, ..., m. \tag{2.6}$$

Hence $y_1 = y_2 = \cdots = y_m$.

To demonstrate the above equalities we substract equalities (2.6) for fixed $j, u \in \{1, 2, ..., m\}$. Consequently by (2.6),

$$-(m-2)\, y_j = \sum_{i=1}^{n-m} |y_{i+m}|. \tag{2.7}$$

If $n = m$, $y \equiv 0$, a contradiction. If $m < n$, then

$$\begin{aligned} 0 &= \sum_{i=1}^{n} f_i y_i = \sum_{i=1}^{m} f_i y_i + \sum_{i=1}^{n-m} f_{i+m} y_{i+m} \\ &= y_1 \cdot a_m + \sum_{i=1}^{n-m} f_{i+m} y_{i+m} \leqslant y_1 a_m + \sum_{i=1}^{n-m} |f_{i+m} y_{i+m}| \\ &\leqslant y_1 \cdot a_m + \sum_{i=1}^{n-m} |y_{i+m}| < y_1(m-2) + \sum_{i=1}^{n-m} |y_{i+m}| = 0 \end{aligned}$$

Since, by (2.7), $y_1 < 0$ and $a_m > m - 2$, we may exclude Case 2. The lemma is proved. ∎

LEMMA 2.4. *Let $f \in S(l_\infty^n)$ satisfy* (2.2) *and let $f_2 < 1$. Let $m \in \{3, ..., n\}$ fulfill $a_m < m - 2$, $a_{m-1} > m - 3$, $f_m > 0$. If there exists $y \in \operatorname{Ker} f \setminus \{0\}$ satisfying the system of inequalities*

$$\begin{cases} y_j \geqslant \displaystyle\sum_{\substack{i=1 \\ i \neq j}}^{m-1} y_i + \sum_{i=1}^{n-m+1} |y_{i+m-1}| & \text{for} \quad j = 2, ..., m-1 \\ y_m \geqslant \displaystyle\sum_{i=1}^{m-1} y_i + \sum_{i=1}^{n-m} |y_{i+m}|, & \end{cases} \tag{2.8}$$

then there exists $y^1 \in \operatorname{Ker} f \setminus \{0\}$ *with*

$$\begin{cases} y_j^1 > \sum_{\substack{i=1 \\ i \neq j}}^{m-1} y_i^1 + \sum_{i=1}^{n-m+1} |y_{i+m-1}^1| & \textit{for} \quad j = 2, ..., m-1 \\ y_m^1 > \sum_{i=1}^{m-1} y_i^1 + \sum_{i=1}^{n-m} |y_{i+m}^1|. \end{cases} \tag{2.9}$$

Proof. Take $y \in \operatorname{Ker} f \setminus \{0\}$ satisfying (2.8) and consider three cases.

Case 1.

$$y_m > \sum_{i=1}^{m-1} y_i + \sum_{i=1}^{n-m} |y_{i+m}|;$$

Then we can select $\theta > 0$ with

$$y_m > \sum_{i=1}^{m-1} y_i + \sum_{i=1}^{n-m} |y_{i+m}| - \theta + \frac{(m-2)\theta}{a_{m-1} - 1}.$$

Put $y_1^1 = y_1 - \theta$, $y_j^1 = y_j + (\theta/a_{m-1} - 1)$ $(j = 2, ..., m-1)$, $y_j^1 = y_j$ for $j = m, ..., n$. Observe that

$$f(y^1) = \sum_{i=1}^{n} f_i y_i^1 = y_1 - \theta + \sum_{i=2}^{m-1} f_i \left(y_i + \frac{\theta}{a_{m-1} - 1} \right) + \sum_{i=m}^{n} f_i y_i$$
$$= \sum_{i=1}^{n} f_i y_i = 0.$$

Since $a_{m-1} > m-3$, $\theta/a_{m-1} - 1 > -\theta + (m-3)(\theta/a_{m-1} - 3)$. Combining this inequality with (2.8), we get

$$y_j^1 > \sum_{\substack{i=1 \\ i \neq j}}^{m-1} y_i^1 + \sum_{i=1}^{n-m+1} |y_{i+m-1}^1| \qquad \text{for} \quad j = 2, ..., m-1$$

$$y_m^1 > \sum_{i=1}^{m-1} y_i^1 + \sum_{i=1}^{n-m} |y_{i+m}^1|,$$

which proves our claim.

Case 2. There exists $j \in \{2, ..., m-1\}$ with

$$y_j > \sum_{\substack{i=1 \\ i \neq j}}^{m-1} y_i + \sum_{i=1}^{n-m+1} |y_{i+m-1}|;$$

Hence

$$y_j - f_j^{-1}\theta > \sum_{\substack{i=1\\ i\neq j}}^{m-1} y_i + \sum_{i=1}^{n-m+1} |y_{i+m-1}| + \theta,$$

for $\theta > 0$ sufficiently small. Since $f_2 < 1$,

$$\theta - f_j^{-1}\theta < 0 \tag{2.10}$$

Put $y^1 = (y_1^1, ..., y_n^1)$, where $y_1^1 = y_1 + \theta$, $y_j^1 = y_j - f_j^{-1}\theta$, $y_i^1 = y_i$ for $i \neq 1, j$. It is clear that $y^1 \in \operatorname{Ker} f$. Adding (2.8) to (2.10), we get for each $k \in \{2, ..., m-1\} \setminus \{j\}$

$$y_k^1 > \sum_{\substack{i=1\\ i\neq k}}^{m-1} y_i^1 + \sum_{i=1}^{n-m+1} |y_{i+m-1}^1|$$

and

$$y_m^1 > \sum_{i=1}^{m-1} y_i^1 + \sum_{i=1}^{n-m} |y_{i+m}^1|,$$

which completes the proof of this case.

Case 3.

$$y_j = \sum_{\substack{i=1\\ i\neq j}}^{m-1} y_i + \sum_{i=1}^{n-m+1} |y_{i+m-1}| \quad \text{for} \quad j = 2, ..., m-1 \tag{2.11}$$

$$y_m = \sum_{i=1}^{m-1} y_i + \sum_{i=1}^{n-m} |y_{i+m}|. \tag{2.12}$$

First we demonstrate that $y_m > 0$. If no, then by (2.11) for every $j \in \{2, ..., m-1\}$

$$y_j = \sum_{\substack{i=1\\ i\neq j}}^{m-1} y_i + \sum_{i=2}^{n-m+1} |y_{i+m-1}| - y_m. \tag{2.13}$$

Substracting inequalities (2.11) for fixed $j, k \in \{2, ..., m-1\}$ we get $y_2 = y_3 = \cdots = y_{m-1}$. According to (2.12) and (2.13) we obtain $y_{m-1} = 0$ which gives $0 = \sum_{i=1}^{n-m+1} |y_{i+m-1}| + y_1$. Since $y \in \operatorname{Ker} f$ and $f_2 < 1$, then $y_{i+m-1} = 0$ for $i = 1, ..., n-m+1$ and consequently $y \equiv 0$, a contradiction. Hence, $y_m > 0$ and reasoning as before we get $y_2 = y_3 = \cdots = y_{m-1}$. Subtracting (2.11) from (2.12) we obtain $y_2 - y_m = y_m - y_2$ and so

$y_2 = y_3 = \cdots = y_m > 0$. Following (2.12) $y_1 = -(m-3)\,y_m - \sum_{i=1}^{n-m} |y_{i+m}|$. Hence,

$$0 = \sum_{i=1}^{n} f_i y_i = \sum_{i=1}^{m} f_i y_i + \sum_{i=m+1}^{n} f_i y_i = -(m-3)\,y_m - \sum_{i=m+1}^{n} |y_i|$$
$$+ (a_m - 1)\,y_m + \sum_{i=m+1}^{n} f_i y_i = (a_m - m + 2)\,y_m + \sum_{i=m+1}^{n} f_i y_i$$
$$- \sum_{i=m+1}^{n} |y_i| < 0,$$

since $a_m < m-2$ and $y_m > 0$. Thus we can exclude Case 3 and the proof of Lemma 2.4 is fully complete.

REMARK 2.5. *Let* $P \in P(X, Y)$, $Y = \operatorname{Ker} f$, *where* f *satisfies* (2.2). *Put for* $i = 1, \ldots, n$ $C_i = \{g \in \operatorname{ext}(X^*)\colon \pm(g \circ P)e_i = \|Pe_i\|\}$. *Then* $g \in \operatorname{crit} P$ (see (0.2)) *if and only if* $g \in \bigcup_{i \in A} C_i$, *where*

$$A = \{i \in \{1, \ldots, n\}\colon \|Pe_i\| = \|P\|\}. \tag{2.14}$$

Proof. Let $g \in \operatorname{crit} P$. Since $\operatorname{ext} X = \{\pm e_i\}_{i=1}^{n}$ $\pm(g \circ P)e_i = \|Pe_i\|$ for some $i \in \{1, \ldots, n\}$. It is clear that $i \in A$. The reverse is obvious.

LEMMA 2.6. *Let* $f \in S(X^*)$ *satisfy* (2.2) *and let* $P_0 \in P(X, Y)$ *is a minimal projection determined by* (0.10) *and* (0.11) *for* $u = u(f)$ *(see Corollary* 0.14); *then*:

(a) *If* $a_u > u-2$ *and* $u = i(f)$ *(we write* $u = u(f)$ *for brevity) then* $A = \{1, \ldots, u\}$. *If* $a_u > u-2$ *and* $u = m(f)$ *then* $A = \{1, \ldots, L\}$ *where* $L = \max\{i \geq m+1\colon f_i^{-1} = \beta_m\}$ *(we write* $m = m(f)$; *see* (0.13)).

(b) *If* $f_2 < 1$, $a_{u-1} > u-3$ *and* $a_u < u-2$, *then* $A = \{2, \ldots, L\}$ *where* $L = \max\{i \geq u\colon f_i = f_u\}$.

Proof. (a) Let y^0 be given by (0.10). It is easy to verify that

$$\|y^0\| = v\beta_u \tag{2.15}$$

and that the following system of inequalities is consistent:

$$\begin{cases} 1 + f_j(\|y^0\| - 2y_j^0) = 1 + v = \|P_0\| & \text{for } j = 1, \ldots, u \\ 1 + f_j\,\|y^0\| \leq 1 + v = \|P_0\| & \text{for } j \geq u+1. \end{cases} \tag{2.16}$$

According to Remark 0.11 and (2.15), we get the desired result.

(b) Let y^0 be given by (0.11). As in the previous case it is easy to check that

$$\|y^0\| = f_u^{-1} v \tag{2.17}$$

and that the following system of inequalities is consistent:

$$\begin{cases} 1 + f_j(\|y^0\| - 2y_j^0) = 1 + v = \|P_0\| & \text{for } j = 2, ..., u \\ 1 + f_j \|y^0\| \leqslant 1 + v = \|P_0\| & \text{for } j \geqslant u + 1 \end{cases} \tag{2.18}$$

By Remark 0.11 we get the desired result.

Now we are able to prove the main result of this section.

THEOREM 2.7. *Let $f \in S(X^*)$ satisfy* (2.2) *and let $u = u(f)$ be given by* (0.12).

$$\textit{If } a_u > u - 2 \tag{2.19}$$

or

$$\textit{if } f_2 < 1, \qquad a_{u-1} > u - 3, \qquad \textit{and} \qquad a_u < u - 2, \tag{2.20}$$

then the projection given by (0.10) *and* (0.11) *is a SUM projection.*

Proof. Let us consider a function $f: S_Y \to \mathbb{R}$ given by

$$\phi(y) = \min\{f_{k(g)} g(y): g \in C\}, \tag{2.21}$$

where

$$C = \{g \in \text{crit } P_0: g(P_0 e_i) = \|P_0\| \text{ for some } i \in \{1, ..., n\}\} \tag{2.22}$$

and

$$k(g) = \min\{i \in \{1, ..., n\}: g(P_0 e_i) = \|P_0\|\}. \tag{2.23}$$

Assume that we can prove that for every $y \in S_Y$ $\phi(y) < 0$. Hence by the compactness of S_Y the constant $\gamma = \sup\{\phi(y): y \in S_Y\}$ is strictly negative. We will prove that P_0 is a SUM projection with $r = -\gamma$. To do this, according to Theorem 0.8(b) and Remark 0.3 it is enough to demonstrate that for every $P \in P(X, Y)$ there exists $g \in C$ (It is clear that $C \cup -C =$ crit P_0 and $C \cap -C = \phi$) with

$$\inf\{g(P - P_0)e_i: e_i \in A_g\} \leqslant -r \|P - P_0\|$$

(see (0.3)). So fix $P \in P(X, Y)$ and let $P - P_0 = f(\cdot)y$ for some $y \in Y$ (we may assume $y \neq 0$). Select $g \in C$ with $f_{k(g)} g(y/\|y\|) = \phi(y/\|y\|)$.

Note that for every $e_i \in A_g$, we have $g(P-P_0)e_i = f_i\, g(y/\|y\|)\,\|y\| \geqslant f_{k(g)}g(y/\|y\|)\,\|y\|$ since $\phi(y/\|y\|)<0$. Hence,

$$\inf\{g(P-P_0)\,e_i : e_i \in A_g\}$$
$$= f_{k(g)}g(y) = \phi(y/\|y\|)\,\|y\| \leqslant \gamma\,\|y\| = -r\,\|P-P_0\|,$$

which, according to Theorem 0.8(b) gives our assertion.

To complete the proof, it suffices to show that $\phi(y)<0$ for every $y \in S_Y$. By (2.22), $k(g)\in A$ for every $g \in C$ (see (2.14)). By Remark 0.11, $f_{k(g)}>0$. Hence accordingly to (2.21), it is enough to verify that for every $y \in S_Y$ $\inf\{g(y) : g \in C\}<0$. By contradiction, assume that there exists $y \in S_Y$ with $g(y)\geqslant 0$ for every $g\in C$ and consider two cases.

Case 1. $a_u > u-2$. If $u=i(f)$ then, following Lemma 2.6(a), the set corresponding to P_0 $A=\{1,...,u\}$. Consequently, by Remark 2.5 and (2.22), $C=\bigcup_{i=1}^n D_i$ where $D_i=\{g\in \operatorname{ext} X^* : (g\circ P_0)\,e_i = \|P_0\|\}$. By Remark 0.11,

$$D_i = \{(-1,...,-1,1_i,-1,...,-1_u,\varepsilon_1,...,\varepsilon_{n-u}) :$$
$$\varepsilon=(\varepsilon_1,...,\varepsilon_{n-u})\in \operatorname{ext} l_\infty^{n-u}\}.$$

Hence the inequalities $g(y)\geqslant 0$ for every $g\in C$ give system (2.3). According to Lemma 2.3, we may find $y^1\in S_Y$ with $g(y^1)>0$ for every $g\in C$. Hence for every $g\in C$ and $e_i\in A_g$

$$f(e_i)\,g(y^1)>0 \qquad \text{since} \quad i\leqslant n \quad \text{and} \quad f_u>0. \tag{2.24}$$

Now define $P=P_0+f(\cdot)\,y^1$ and note that (2.24) implies

$$\inf\{g(P-P_0)\,e_i : e_i\in A_g\}>0. \tag{2.25}$$

According to Theorem 0.8(a) and Remark 0.9, P_0 is not a minimal projection, which contradicts Theorem 0.10. If $u(f)<i(f)$, then the set A is equal to $\{1,...,L\}$ where L is given in Lemma 2.6. Hence $C=U_{i=1}^n D_i$ where D_i for $i=1,2,...,u$ are as above and for $i\geqslant u+1$. $D_i=\{(-1,...,-1,1_u,\varepsilon_1,...,\varepsilon_{i-1},1_i,\varepsilon_i,...,\varepsilon_{n-u-1}) : \varepsilon\in \operatorname{ext}(l_\infty^{n-u-1})\}$. So to the system (2.3) we must add the system

$$y_j \geqslant \sum_{i=1}^{u} y_i + \sum_{\substack{i=1\\ i\neq j}}^{n-u} |y_{i+u}| \qquad \text{for} \quad j=u+1,...,L.$$

By Lemma 2.3, there exists

$$y^1 \in \operatorname{Ker} f$$

with

$$y_j^1 > \sum_{\substack{i=1 \\ i \neq j}}^{u} y_i^1 + \sum_{i=1}^{n-u} |y_{i+u}^1| \qquad \text{for} \quad j = 1, 2, ..., u.$$

Now replace f by $f^1 = (1, f_2, ..., f_u, f^1_{u+1}, ..., f^1_n)$ where $f_{u+1} > f^1_{u+1} \geqslant \cdots \geqslant f^1_n$. Note that in view of Theorem 0.10 the operator

$$P_0^1 = I - f^1(\cdot)\, y^0 \tag{2.26}$$

(y^0 is the vector from X corresponding to P_0) is a minimal projection onto $\operatorname{Ker} f^1$. If the change of f_{u+1} is slight, then modifying slightly the $n-u$ last coordinates of vector y^1 we get $y^2 = (y^1_1, ..., y^1_u, y^2_{u+1}, ..., y^2_n) \in \operatorname{Ker} f^1$ satisfying (2.4). Since $\beta_u < 1/f^1_{u+1}$, reasoning as in the previous situation by Theorem 0.8(a) we get that P_0^1 is not a minimal projection onto $\operatorname{Ker} f^1$; this is a contradiction.

Case 2. $a_u < u-2, a_{u-1} > u-3, f_2 < 1$. Since $a_u < u-2$, by (0.8), $u = i(f)$. If $f_{u+1} < f_u$, according to Lemma 2.6,

$$A = \{2, ..., u\} \qquad \text{and} \qquad C = \bigcup_{i=2}^{u} D_i$$

where, in view of Remark 0.11,

$$D_i = \{(-1, ..., -1, 1_i, -1, ..., -1_{u-1}, \varepsilon_1, ..., \varepsilon_{n-u+1}) : \varepsilon \in \operatorname{ext}(l_\infty^{n-u+1})\} \qquad \text{for} \quad i = 2, ..., u-1$$

and

$$D_u = \{(-1, ..., -1, 1_u, \varepsilon_1, ..., \varepsilon_{n-u}) : \varepsilon \in \operatorname{ext}(l_\infty^{n-u})\}.$$

Hence the inequalities $g(y) \geqslant 0$ for every $g \in C$ form system (2.8). By Lemma 2.4 there exists $y^1 \in Y$ with $g(y^1) > 0$ for every $g \in C$. Reasoning as in Case 1, we get a contradiction with the minimality of P_0.

If $f_{u+1} = f_u$, then $C = \bigcup_{i=2}^{L} D_L$, where L is defined in Lemma 2.6 and

$$D_i = \{(-1, ..., -1, -1_{u-1}, \varepsilon_1, ..., \varepsilon_{i-1}, 1_i, \varepsilon_i, ..., \varepsilon_{n-u}): \varepsilon \in \operatorname{ext}(l_\infty^{n-u})\} \quad \text{for} \quad i \geqslant u$$

(for $i = 2, ..., u$ the sets D_i are defined as before). Hence to the system (2.8) we must add the following inequalities:

$$y_j \geqslant \sum_{i=1}^{u-1} y_i + \sum_{\substack{i=1 \\ i \neq j}}^{n-u} |y_{i+u-1}| \qquad \text{for} \quad j \geqslant u+1.$$

According to Lemma 2.4, there exists $y^1 \in Y$ with

$$y_j^1 > \sum_{\substack{i=1 \\ i \neq j}}^{u-1} y_i^1 + \sum_{i=1}^{n-u+1} |y_{i+u-1}^1| \qquad \text{for} \quad j = 2, ..., u-1$$

and

$$y_u^1 > \sum_{i=1}^{u-1} y_i^1 + \sum_{i=2}^{n-u+1} |y_{i+u-1}^1|.$$

Modifying, as in Case 1, f onto f^1, where

$$f^1 = (f_1, ..., f_u, f_{u+1}^1, ..., f_u^1), f_{u+1}^1 < f_u,$$

and y^1 to y^2 belonging to $\operatorname{Ker} f^1$, we get a contradiction as in Case 1. The proof of Theorem 2.7 is fully complete ∎.

In [9] it was shown by a different method that the conditions (2.19) and (2.20) are equivalent to the unicity of minimal projection. Combining this with Proposition 2.1 and Theorem 2.7 we get

THEOREM 2.8 *Let $P_0 \in P(X, Y)$ be a minimal projection. Then P_0 is a unique minimal projection if and only if P_0 is a SUM projection.*

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